Introductory Notes on Fractional Calculus

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July 31, 2006

Abstract

Our understanding of Nature relies on calculus, which in turn relies on the intuitive concept of the derivative. It's descriptive power comes from the fact that it analyses the behavior at scales small enough that its properties change linearly, so avoiding complexities that arise at larger ones. Fractional Calculus generalizes this concept from integer to non-integer order. Despite it seems not to have significant applications in fundamental physics, research on this core concept could be valuable in understanding Nature. These notes comprise an introduction to the field.

1 Introduction

Fractional Calculus is the branch of calculus that generalizes the derivative of a function to non-integer order, allowing calculations such as deriving a function to 1/2 order. Despite "generalized" would be a better option, the name "fractional" is used for denoting this kind of derivative.

!*Note:* These notes are not a summary of standard methods or an academic text, so some conventional developments are omitted while some other are introduced. The same happens with notation.

The derivative of a function f is defined as

$$D^1 f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{1.1}$$

Iterating this operation yields an expression for the n-st derivative of a function. As can be easily seen -and proved by induction- for any natural number n,

$$D^n f(x) = \lim_{h \to 0} h^{-n} \sum_{m=0}^{n} (-1)^m \binom{n}{m} f(x + (n-m)h) \tag{1.2}$$

where

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!} \tag{1.3}$$

or equivalently,

$$D^n f(x) = \lim_{h \to 0} h^{-n} \sum_{m=0}^{n} (-1)^m \binom{n}{m} f(x - mh) \tag{1.4}$$

The case of $n = 0$ can be included as well.

Such an expression could be valuable for instance in a simple program for plotting the n-st derivative of a function.

Viewing this expression one asks immediately if it can be generalized to any non-integer, real or complex number n. There are some reasons that can make us think so,

1. The fact that for any natural number n the calculation of the n-st derivative is given by an explicit formula (1.2) or (1.4).
2. That the generalization of the factorial by the gamma function allows

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!} = \frac{\Gamma(n+1)}{\Gamma(m+1)\,\Gamma(n-m+1)} \tag{1.5}$$

   which also is valid for non-integer values.
3. The likeness of (1.2) to the binomial formula

$$(a+b)^n = \sum_{m=0}^{n} \binom{n}{m} a^{n-m} b^m \tag{1.6}$$

   which can be generalized to any complex number α by

$$(a+b)^\alpha = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1)}{n!\,\Gamma(\alpha-n+1)} a^{\alpha-n} b^n \tag{1.7}$$

   which is convergent if

$$|\,b\,| < a \tag{1.8}$$

There are some desirable properties that could be required to the fractional derivative,

1. Existance and continuity for m times derivable functions, for any n which modulus is equal or less than m.

2. For $n = 0$ the result should be the function itself; for $n > 0$ integer values it should be equal to the ordinary derivative and for $n < 0$ integer values it should be equal to ordinary integration -regardless the integration constant.

3. Iterating should not give problems,

$$D^{\alpha+\beta} f(x) = D^{\alpha} D^{\beta} f(x) \tag{1.9}$$

4. Linearity,

$$D^{\alpha}(a f(x) + b g(x)) = a D^{\alpha} f(x) + b D^{\alpha} g(x) \tag{1.10}$$

5. Allowing Taylor's expansion in some other way.

6. Its characteristic property should be preserved for the exponential function,

$$D^{\alpha} e^{x} = e^{x} \tag{1.11}$$

2 Exponentials

The case of the exponential function is specially simple and gives some clues about the generalization of the derivatives. Following (1.2),

$$\begin{aligned} D^{\alpha} e^{ax} &= \lim_{h\to 0} h^{-\alpha} \sum_{n=0}^{\alpha} (-1)^n \binom{\alpha}{n} e^{a(x+(\alpha-n)h)} = \\ &= e^{ax} \lim_{h\to 0} h^{-\alpha} \sum_{n=0}^{\alpha} (-1)^n \binom{\alpha}{n} (e^{ah})^{\alpha-n} = \\ &= e^{ax} \lim_{h\to 0} h^{-\alpha} (e^{ah} - 1)^{\alpha} = \\ &= a^{\alpha} e^{ax} \end{aligned} \tag{2.1}$$

the above limit exists for any complex number α. However, it should be noted that in the substitution of the binomial formula a natural number has been considered. We shall deal with this problem later to get our first generalization of the derivative. Applying this to the imaginary unit,

$$\begin{aligned} D^{\alpha} \cos(x) + i\, D^{\alpha} \sin(x) &= D^{\alpha} e^{ix} = i^{\alpha} e^{ix} = \\ &= e^{\frac{\alpha\pi i}{2}} e^{ix} = e^{i(x+\frac{\alpha\pi}{2})} = \\ &= \cos\left(x + \frac{\alpha\pi}{2}\right) + i \sin\left(x + \frac{\alpha\pi}{2}\right) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} D^\alpha \cos(x) - i\, D^\alpha \sin(x) =&\ D^\alpha\, e^{-ix} = (-i)^\alpha e^{-ix} = \\ &= e^{\frac{-\alpha\pi i}{2}} e^{-ix} = e^{-i(x+\frac{\alpha\pi}{2})} = \\ &= \cos\left(x + \frac{\alpha\pi}{2}\right) - i \sin\left(x + \frac{\alpha\pi}{2}\right) \end{aligned} \tag{2.3}$$

solving this system we have the next definition for the sine and cosine derivatives,

$$D^\alpha \sin(x) = \sin\left(x + \frac{\alpha\pi}{2}\right) \tag{2.4}$$

and

$$D^\alpha \cos(x) = \cos\left(x + \frac{\alpha\pi}{2}\right) \tag{2.5}$$

We could expect these relations for the sine and cosine derivatives to be maintained in the generalization of the derivative.

Applying the above method we also can calculate the following,

$$\begin{aligned} D^\alpha \cos(ax) + i\, D^\alpha \sin(ax) =&\ D^\alpha\, e^{iax} = (ai)^\alpha e^{iax} = \\ &= a^\alpha e^{\frac{\alpha\pi i}{2}} e^{iax} = a^\alpha e^{i(ax+\frac{\alpha\pi}{2})} = \\ &= a^\alpha \cos\left(ax + \frac{\alpha\pi}{2}\right) + i a^\alpha \sin\left(ax + \frac{\alpha\pi}{2}\right) \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} D^\alpha \cos(ax) - i\, D^\alpha \sin(ax) =&\ D^\alpha\, e^{-iax} = (-ai)^\alpha e^{-iax} = \\ &= a^\alpha e^{-\frac{\alpha\pi i}{2}} e^{-iax} = a^\alpha e^{-i(ax+\frac{\alpha\pi}{2})} = \\ &= a^\alpha \cos\left(ax + \frac{\alpha\pi}{2}\right) - i a^\alpha \sin\left(ax + \frac{\alpha\pi}{2}\right) \end{aligned} \tag{2.7}$$

Thus,

$$D^\alpha \sin(ax) = a^\alpha \sin\left(ax + \frac{\alpha\pi}{2}\right) \tag{2.8}$$

and

$$D^\alpha \cos(ax) = a^\alpha \cos\left(ax + \frac{\alpha\pi}{2}\right) \tag{2.9}$$

Indeed, the above result of the exponential can be applied to any function that can be expanded in exponentials

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx} \implies$$

$$\implies D^\alpha f(x) = \sum_{n=-\infty}^{\infty} a_n D^\alpha e^{inx} = \sum_{n=-\infty}^{\infty} a_n (ni)^\alpha e^{inx} =$$

$$= \sum_{n=-\infty}^{\infty} a_n n^\alpha e^{\frac{\alpha\pi i}{2}} e^{inx} = \sum_{n=-\infty}^{\infty} a_n n^\alpha e^{i(nx+\frac{\alpha\pi}{2})} \tag{2.10}$$

Expanding the function in Fourier series,

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{e^{inx}}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt \implies$$

$$\implies D^\alpha f(x) = \sum_{n=-\infty}^{\infty} \frac{n^\alpha e^{i(nx+\frac{\alpha\pi}{2})}}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt \tag{2.11}$$

This method can be useful for calculating fractional derivatives of trigonometric functions.

3 Powers

The case of powers of x also has some simplicity that allows its generalization. The case of integer order derivatives

$$D^1 x^a = a x^{a-1} \implies D^n x^a = x^{a-n} \prod_{m=0}^{n-1} (a-m) = \frac{a!}{(a-n)!} x^{a-n} \tag{3.1}$$

can be easily generalized to non-integer order derivatives

$$D^\alpha x^a = \frac{\Gamma(a+1)}{\Gamma(a-\alpha+1)} x^{a-\alpha} \tag{3.2}$$

which can be applied to any function that can be expanded in powers of x

$$f(x) = \sum_{n=-\infty}^{\infty} a_n x^n \implies$$

$$\implies D^\alpha f(x) = \sum_{n=-\infty}^{\infty} a_n D^\alpha x^n = \sum_{n=-\infty}^{\infty} a_n \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha} \tag{3.3}$$

Expanding the function in Taylor series,

$$f(x+a) = \sum_{n=0}^{\infty} \frac{D^n f(a)}{n!} x^n \implies$$
$$\implies D^\alpha f(x+a) = \sum_{n=0}^{\infty} \frac{D^n f(a)}{\Gamma(n-\alpha+1)} x^{n-\alpha} \tag{3.4}$$

or expanding it in Laurent series,

$$f(x+a) = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} x^n \int_C \frac{f(t+a)}{t^{n+1}} dt \implies$$
$$\implies D^\alpha f(x+a) = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha} \int_C \frac{f(t+a)}{t^{n+1}} dt \tag{3.5}$$

This can be an useful tool for calculating fractional derivatives. However, we should compare these results of powers with the previous results of exponentials to see if they agree. With the result of exponentials (2.1),

$$D^\alpha e^{ax} = a^\alpha e^{ax} = a^\alpha \sum_{n=0}^{\infty} \frac{(ax)^n}{n!} = \sum_{n=0}^{\infty} \frac{a^{n+\alpha} x^n}{n!} \tag{3.6}$$

but with the result of powers (3.4),

$$D^\alpha e^{ax} = D^\alpha \sum_{n=0}^{\infty} \frac{(ax)^n}{n!} = \sum_{n=0}^{\infty} \frac{a^n x^{n-\alpha}}{\Gamma(n-\alpha+1)} \tag{3.7}$$

If we compare both results, we see that they only agree for integer values of α. We shall see later where these discrepancies come from, and how they can be avoided.

4 Binomial Formula

In the expression (1.2) the exponentian function allows the substitution of the binomial formula as done in (2.1), but this is not possible for any given function. For applying this substitution we require the following displacement operator,

$$d_h f(x) = f(x+h) \tag{4.1}$$

whose iteration yields

$$d_h^\alpha f(x) = f(x+\alpha h) \tag{4.2}$$

what allows the application of the binomial formula (1.6) for natural numbers and the generalized binomial formula (1.7) for complex numbers,

$$\begin{aligned} D^n f(x) &= \lim_{h\to 0} h^{-n} \sum_{m=0}^{n} (-1)^m \binom{n}{m} f(x+(n-m)h) = \\ &= \lim_{h\to 0} h^{-n} \sum_{m=0}^{n} (-1)^m \binom{n}{m} d_h^{n-m} f(x) = \\ &= \lim_{h\to 0} \left(\frac{d_h - 1}{h}\right)^n f(x) \end{aligned} \tag{4.3}$$

so that for any complex number α it can be generalized

$$D^\alpha f(x) = \lim_{h\to 0} \left(\frac{d_h - 1}{h}\right)^\alpha f(x) \tag{4.4}$$

This sheds more light on the derivative and its generalization. Using the expression of the generalized binomial formula (1.7) for non-integer numbers,

$$\begin{aligned} D^\alpha f(x) &= \lim_{h\to 0} \left(\frac{d_h - 1}{h}\right)^\alpha f(x) = \\ &= \lim_{h\to 0} h^{-\alpha} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1)}{n!\,\Gamma(\alpha-n+1)} (-1)^n d_h^{\alpha-n} f(x) = \\ &= \lim_{h\to 0} h^{-\alpha} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\alpha+1)}{n!\,\Gamma(\alpha-n+1)} f(x+(\alpha-n)h) \end{aligned} \tag{4.5}$$

In the case of integer values the summatory only extends α terms and it is equal to the ordinary derivative.

Finally, it is obvious that as h goes to 0 the last equation is equivalent to the following

$$D^\alpha f(x) = \lim_{h\to 0} h^{-\alpha} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\alpha+1)}{n!\,\Gamma(\alpha-n+1)} f(x-nh) \tag{4.6}$$

5 Functions of the Derivative

It is worth to mention that being (4.4) the expression of the α-st derivative of a function the derivative itself is

$$D = \lim_{h\to 0} \frac{d_h - 1}{h} \tag{5.1}$$

And being the α-st derivative or iterating the differentiation α times powering it to α, applying other functions to the derivative could be also considered. If the function applied to the derivative can be expanded in powers of x,

$$g(x) = \sum_{n=-\infty}^{\infty} a_n x^n \implies g(D) f(x) = \sum_{n=-\infty}^{\infty} a_n D^n f(x) \tag{5.2}$$

The result is a weighted sum of different order derivatives. These functions of the derivative are usually known as “formal differential operators”. As an example, the exponential of the derivative applied to the exponential would give the following result that could be valuable for calculating functions of the derivative when both f and g can be expanded in exponentials

$$\begin{aligned} e^{ax} &= \sum_{n=0}^{\infty} \frac{a^n x^n}{n!} \implies \\ &\implies e^{aD} e^{bx} = \sum_{n=0}^{\infty} \frac{a^n D^n e^{bx}}{n!} = \sum_{n=0}^{\infty} \frac{a^n b^n e^{bx}}{n!} = e^{ab} e^{bx} = e^{b(x+a)} \end{aligned} \tag{5.3}$$

If both functions f and g can be expanded in positive powers of x,

$$\begin{aligned} g(x) &= \sum_{n=0}^{\infty} a_n x^n \quad , \quad f(x) = \sum_{n=0}^{\infty} b_n x^n \implies \\ &\implies g(D) f(x) = \sum_{n=0}^{\infty} a_n D^n \sum_{m=0}^{\infty} b_m x^m = \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} b_m D^n x^m = \\ &= \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} b_m \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n} = \\ &= \sum_{n=0}^{\infty} a_n \sum_{l=-n}^{\infty} b_{l+n} \frac{\Gamma(l+n+1)}{\Gamma(l+1)} x^l = \\ &= \sum_{n=0}^{\infty} a_n \sum_{l=0}^{\infty} b_{l+n} \frac{(l+n)!}{l!} x^l = \\ &= \sum_{l=0}^{\infty} \frac{x^l}{l!} \sum_{n=0}^{\infty} (l+n)!\, a_n b_{l+n} \end{aligned} \tag{5.4}$$

Thus,

$$\begin{aligned} g(D) f(x) &= \sum_{n=0}^{\infty} c_n x^n \\ c_n &= \frac{1}{n!} \sum_{m=0}^{\infty} (n+m)!\, a_m b_{n+m} \end{aligned} \tag{5.5}$$

At a first glance this seems not very interesting, but interesting properties could be hidden under this apparent mess. We shall return later to this point with the help of other tools.

6 Grunwald-Letnikov Derivative

Grunwald-Letnikov derivative or also named Grunwald-Letnikov differintegral, is a generalization of the derivative analogous to our generalization by the binomial formula (4.6), but it is based on the direct generalization of the equation (1.4). The idea behind is that h should approach 0 as n approaches infinity,

$$\begin{aligned} D^n f(x) &= \lim_{h\to 0} h^{-n} \sum_{m=0}^{n} (-1)^m \binom{n}{m} f(x-mh) = \\ &= \lim_{h\to 0} h^{-n} \sum_{m=0}^{n} (-1)^m \frac{n!}{m!\,\Gamma(n-m+1)} f(x-mh) = \\ &= \lim_{h\to 0} h^{-n} \sum_{m=0}^{\infty} (-1)^m \frac{n!}{m!\,\Gamma(n-m+1)} f(x-mh) = \\ &= \lim_{h\to 0} h^{-n} \sum_{m=0}^{\frac{x-a}{h}} (-1)^m \frac{n!}{m!\,\Gamma(n-m+1)} f(x-mh) \end{aligned} \tag{6.1}$$

being a a negative infinity. However, the generalization is done so that any wished a less than x can be chosen. The above equation can be generalized now to get a formula equivalent to (4.6)

$$D^\alpha f(x) = \lim_{h\to 0} h^{-\alpha} \sum_{m=0}^{\frac{x-a}{h}} (-1)^m \frac{\Gamma(\alpha+1)}{m!\,\Gamma(\alpha-m+1)} f(x-mh) \tag{6.2}$$

or equivalently,

$$D^\alpha f(x) = \lim_{n\to\infty} \left(\frac{n}{x-a}\right)^\alpha \sum_{m=0}^{n} (-1)^m \frac{\Gamma(\alpha+1)}{m!\,\Gamma(\alpha-m+1)} f\left(x - m\left(\frac{x-a}{n}\right)\right) \tag{6.3}$$

These generalizations have the inconvenient that if α is a negative integer they are not valid for the gamma function is infinity in negative integers and zero. However, for negative α values the following can be used

$$\begin{aligned} \binom{-n}{m} &= \frac{\prod_{l=-n-m+1}^{-n} l}{m!} = \frac{\prod_{l=n}^{n+m-1} (-l)}{m!} = \frac{(-1)^m \prod_{l=n}^{n+m-1} l}{m!} = \\ &= \frac{(-1)^m (n+m-1)!}{m!\,(n-1)!} = \frac{(-1)^m\,\Gamma(n+m)}{m!\,\Gamma(n)} \end{aligned} \tag{6.4}$$

so that for negative values equations (6.2) and (6.3) can be changed to

$$D^{-\alpha} f(x) = \lim_{h\to 0} h^{-\alpha} \sum_{m=0}^{\frac{x-a}{h}} \frac{\Gamma(\alpha+m)}{m!\,\Gamma(\alpha)} f(x-mh) \tag{6.5}$$

and

$$D^{-\alpha} f(x) = \lim_{n\to\infty} \left(\frac{n}{x-a}\right)^{\alpha} \sum_{m=0}^{n} \frac{\Gamma(\alpha+m)}{m!\,\Gamma(\alpha)}\, f\left(x - m\left(\frac{x-a}{n}\right)\right) \tag{6.6}$$

The reason why $x-a$ is considered in the above equations instead of a single positive number can be seen considering the case of $\alpha = -1$, which must give the same result that integrating the function,

$$\begin{aligned}
\int_a^x f(t)\,dt = D^{-1} f(x) &= \lim_{\substack{n\to\infty \\ h\to 0}} h \sum_{m=0}^{n} \frac{\Gamma(m+1)}{m!\,\Gamma(1)}\, f(x-mh) = \\
&= \lim_{\substack{n\to\infty \\ h\to 0}} \sum_{m=0}^{n} h f(x-mh) = \lim_{\substack{n\to\infty \\ h\to 0}} \int_0^{nh} f(x-t)\,dt = \\
&= \lim_{\substack{n\to\infty \\ h\to 0}} \int_{x-nh}^{x} f(t)\,dt \implies \\
&\implies a = x - nh \implies n = \frac{x-a}{h} \quad , \quad h = \frac{x-a}{n}
\end{aligned} \tag{6.7}$$

Yes, in fractional derivatives limits must be considered as they must be in integrals. Limits only vanish with integer order derivatives. This also means that fractional derivatives are nonlocal, which may be the reason that makes this kind of derivatives less useful in describing Nature. We shall see later more about the integration -or better say differentiation- limits.

In the case of the derivative generalized by the binomial formula (4.6), since n goes to infinity regardless of h, $x-a$ must be infinity, so that the derivative defined in (4.4) and (4.6) is equivalent to the Grunwald-Letnikov derivative with a lower limit of negative infinity.

7 Riemann-Liouville Derivative

Riemann-Liouville derivative is the most used generalization of the derivative. It is based on Cauchy's formula for calculating iterated integrals. If the first integral of a function, which must equal to deriving it to -1, is as follows

$$D^{-1} f(x) = \int_0^x f(t)\,dt \tag{7.1}$$

the calculation of the second can be simplified by interchanging the integration order

$$D^{-2} f(x) = \int_0^x \int_0^{t_2} f(t_1)\, dt_1\, dt_2 = \int_0^x \int_{t_1}^x f(t_1)\, dt_2\, dt_1 = \int_0^x f(t_1) \int_{t_1}^x dt_2\, dt_1 =$$
$$= \int_0^x f(t)\,(x-t)\, dt \tag{7.2}$$

This method can be applied repeatedly, resulting in the following formula for calculating iterated integrals,

$$D^{-n} f(x) = \frac{1}{(n-1)!} \int_0^x f(t)\,(x-t)^{n-1}\, dt \tag{7.3}$$

Now this can be easily generalized to non-integer values, in what is the Riemann-Liouville derivative,

$$D^{\alpha} f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x \frac{f(t)}{(x-t)^{\alpha+1}}\, dt \tag{7.4}$$

Note however that in the above formulas the election of 0 as the lower limit of integration has been arbitrary, and any other number could be chosen. Generally, the election of the integration limits in this and other generalizations of the derivative is indicated with subscripts. The Riemann-Liouville derivative with the lower integration limit a would be

$${}_aD_x^{\alpha} f(x) = \frac{1}{\Gamma(-\alpha)} \int_a^x \frac{f(t)}{(x-t)^{\alpha+1}}\, dt \tag{7.5}$$

The problem with this generalization is that if the real part of α is positive or zero the integral diverges. So it only can be used to calculate generalized integrals. However, this can be solved easily by deriving first by ordinary derivative more than the amount necessary, thus making the remaining necessary differentiation negative and then applying the generalized derivative for completing the rest in which will be a negative differentiation,

$$\begin{aligned} {}_aD_x^{\alpha} f(x) &= D^n\, {}_aD_x^{\alpha-n} f(x) \\ n &= \left[\Re(\alpha)\right] + 1 \end{aligned} \tag{7.6}$$

Once seen how integration limits must be specified in fractional derivatives, we can return to the derivative of the exponential (2.1) and the derivative of powers (3.2) to see how their disagreement (3.7) was caused by having different limits. With the exponential, supposing b a positive number,

$$b^{-1}e^{bx} = {}_aD_x^{-1}e^{bx} = \int_a^x e^{bx}\,dx = \frac{e^{bx}-e^{ba}}{b} \implies \frac{e^{ba}}{b} = 0 \implies$$
$$\implies a = -\infty \tag{7.7}$$

while with the powers

$$\frac{x^{b+1}}{b+1} = {}_aD_x^{-1}x^b = \int_a^x x^b\,dx = \frac{x^{b+1}-a^{b+1}}{b+1} \implies \frac{a^{b+1}}{b+1} = 0 \implies$$
$$\implies a = 0 \tag{7.8}$$

So both results were different in the case of the exponential function because the limits of the derivatives were different. And the case of this discrepancy is not casual. Differences in the results of derivatives with different differentiation limits are indeed important.

Finally, it can be proved that the Grunwald-Letnikov derivative with any given integration limits is equal to the Riemann-Liouville derivative with the same limits for any complex number α with a negative real part. This important result means that our derivative generalized by the binomial formula in (4.4) and (4.6) is equivalent to the Riemann-Liouville derivative with a lower limit of negative infinity provided that the real part of α is negative,

$${}_{-\infty}D_x^{\alpha}f(x) = \lim_{h\to 0}\left(\frac{d_h-1}{h}\right)^{\alpha} f(x) = \frac{1}{\Gamma(-\alpha)}\int_{-\infty}^{x}\frac{f(t)}{(x-t)^{\alpha+1}}\,dt \tag{7.9}$$

This kind of Riemann-Liouville derivative with a lower limit of negative infinity is known as the Weyl derivative.

8 Domain Transforms

The Laplace and Fourier transforms that serve to transform to the frequency domain can be used to get generalizations of the derivative valid for functions that allow such transformations. The Laplace transform is defined by

$$\mathcal{L}\left\{f(x)\right\} = \int_0^{\infty} e^{-tx}f(x)\,dx \tag{8.1}$$

while its inverse transform is

$$\mathcal{L}^{-1}\left\{f(x)\right\} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{tx}f(x)\,dx \tag{8.2}$$

where a is chosen so that it is greather than the real part of any of the singularities of $f(x)$. An important property of the Laplace transform is related to the transform of the n-st derivative of a function,

$$\mathcal{L}\left\{D^n f(x)\right\} = t^n \mathcal{L}\left\{f(x)\right\} - \sum_{m=0}^{n-1} t^m \left(D^{n-m-1} f\right)(0) \tag{8.3}$$

In the case that the terms in the summatory are zero the relation is particularly simple, and for those kind of functions the derivative can be generalized so that this property holds true for non-integer values of α

$$\begin{aligned}(D^l\, {}_0D_x^{\alpha-m} f)(0) = 0 \qquad l = 0\ldots m-1 \qquad m = \left[\Re(\alpha)\right] + 1 \implies \\ \implies \mathcal{L}\left\{D^\alpha f(x)\right\} = t^\alpha \mathcal{L}\left\{f(x)\right\}\end{aligned} \tag{8.4}$$

for which the generalized derivative can be defined as

$$D^\alpha f(x) = \mathcal{L}^{-1}\left\{t^\alpha \mathcal{L}\left\{f(x)\right\}\right\} \tag{8.5}$$

Keeping in mind the result of the generalized derivative of the exponential (2.1), the following development provides an easy understanding of the reasons involved in the above generalization,

$$\begin{aligned} D^\alpha f(x) &= D^\alpha \mathcal{L}^{-1}\left\{\mathcal{L}\left\{f(x)\right\}\right\} = \\ &= D^\alpha \left(\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{tx} \int_0^\infty e^{-tx} f(x)\,dx\,dt\right) = \\ &= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} D^\alpha e^{tx} \int_0^\infty e^{-tx} f(x)\,dx\,dt = \\ &= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} t^\alpha e^{tx} \int_0^\infty e^{-tx} f(x)\,dx\,dt = \\ &= \mathcal{L}^{-1}\left\{t^\alpha \mathcal{L}\left\{f(x)\right\}\right\} \end{aligned} \tag{8.6}$$

On the other hand, the Fourier transform is defined by

$$\mathcal{F}\left\{f(x)\right\} = \int_{-\infty}^{\infty} e^{-itx} f(x)\,dx \tag{8.7}$$

while its inverse transform is

$$\mathcal{F}^{-1}\left\{f(x)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itx} f(x)\,dx \tag{8.8}$$

This transform also has an analogous property related to the transform of the n-st derivative of a function,

$$\mathcal{F}\left\{D^n f(x)\right\} = (it)^n \mathcal{F}\left\{f(x)\right\} \tag{8.9}$$

and the derivative can be generalized so that this property holds true for non-integer values of α

$$\mathcal{F}\left\{D^\alpha f(x)\right\} = (it)^\alpha \mathcal{F}\left\{f(x)\right\} \tag{8.10}$$

yielding the following definition of the generalized derivative

$$D^\alpha f(x) = \mathcal{F}^{-1}\left\{(it)^\alpha \mathcal{F}\left\{f(x)\right\}\right\} \tag{8.11}$$

The application of (2.1) can provide again an easy understanding of the reasons involved in the generalization,

$$\begin{aligned} D^\alpha f(x) &= D^\alpha \mathcal{F}^{-1}\left\{\mathcal{F}\left\{f(x)\right\}\right\} = \\ &= D^\alpha \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itx} \int_{-\infty}^{\infty} e^{-itx} f(x)\, dx\, dt\right) = \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} D^\alpha e^{itx} \int_{-\infty}^{\infty} e^{-itx} f(x)\, dx\, dt = \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (it)^\alpha e^{itx} \int_{-\infty}^{\infty} e^{-itx} f(x)\, dx\, dt = \\ &= \mathcal{F}^{-1}\left\{(it)^\alpha \mathcal{F}\left\{f(x)\right\}\right\} \end{aligned} \tag{8.12}$$

In these two generalizations the implicit limits of differentiation should be determined. In the case of Laplace transform, the generalized derivative is a Riemann-Liouville derivative with a lower limit of 0, whereas in the case of Fourier transform it is a Weyl derivative. Indeed, if we check for instance the derivative generalized by the Fourier transform in the cases of the sine and cosine functions -calculated in (2.4), (2.5), (2.8) and (2.9) with the generalized derivative of the exponential that we have seen in (7.7) that also is a Weyl derivative- we will find that they match perfectly.

9 Convolution

The generalizations of the derivative as expressed in (4.6) and (7.4) suggest that they can be formulated in terms of the convolution, which would be important for the convolution is a very simple operation in the frequency spaces achieved by Laplace and Fourier transforms. The following development shows how this is the case, and how after all the derivative of a function is its convolution with certain function,

$$\Phi_\alpha(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)} \tag{9.1}$$

which Laplace convolution with $f(x)$ yields the Riemann-Liouville derivative of order $-\alpha$

$$\begin{aligned}\Phi_\alpha(x) * f(x) = & \int_0^x \Phi_\alpha(x-t)\, f(t)\, dt = \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(-\alpha)}\, f(t)\, dt = \\ & = D^{-\alpha} f(x)\end{aligned} \tag{9.2}$$

And since the transform of the function expressed in (9.1) gives the following simple result

$$\mathcal{L}\left\{x^\alpha\right\} = \frac{\Gamma(\alpha+1)}{t^{\alpha+1}} \implies \mathcal{L}\left\{\Phi_\alpha\right\} = \mathcal{L}\left\{\frac{x^{\alpha-1}}{\Gamma(\alpha)}\right\} = t^{-\alpha} \tag{9.3}$$

and the transform of the convolution is the multiplication of the transforms, in the case that the function f fulfills all the requirements given previously for the simplicity of the Laplace transform of the derivative, we again get the equation (8.4), which was the key for the generalization by Laplace transform

$$\begin{aligned}\mathcal{L}\left\{D^{-\alpha} f(x)\right\} & = \mathcal{L}\left\{\Phi_\alpha(x) * f(x)\right\} = \mathcal{L}\left\{\Phi_\alpha(x)\right\} \mathcal{L}\left\{f(x)\right\} = \\ & = t^{-\alpha} \mathcal{L}\left\{f(x)\right\}\end{aligned} \tag{9.4}$$

These last results showing that the generalized derivative is a convolution with certain function, open the interesting question of what other kind of operators would be defined if functions other than (9.1) had been chosen. The answer is that the new operators defined would be functions of the derivative. To see this, we can exploit the linearity of the convolution, supposing that the function g can be expanded in powers of x

$$\begin{aligned}
g(x) = \sum_{n=-\infty}^{\infty} a_n x^n \implies \\
\implies g(D)\, f(x) &= \sum_{n=-\infty}^{\infty} a_n\, D^n\, f(x) = \sum_{n=-\infty}^{\infty} a_n\, \Phi_{-n}(x) * f(x) = \\
&= \left(\sum_{n=-\infty}^{\infty} \frac{a_n\, x^{-n-1}}{\Gamma(-n)} \right) * f(x) = \\
&= \left(\sum_{n=-\infty}^{\infty} \frac{a_{-n-1}\, x^{n}}{\Gamma(n+1)} \right) * f(x) = \\
&= h(x) * f(x) \qquad (9.5)
\end{aligned}$$

where

$$\begin{aligned}
h(x) &= \sum_{n=-\infty}^{\infty} b_n\, x^n \\
b_n &= \frac{a_{-n-1}}{\Gamma(n+1)} \qquad (9.6)
\end{aligned}$$

This proves the equivalence between functions of the derivative and convolutions. For any convolution there is an equivalent function of the derivative if the function implied can be expanded in powers of x and vice versa. Now, the Laplace transform shows what was expectable,

$$\begin{aligned}
\mathcal{L}\left\{g(D)\, f(x)\right\} &= \mathcal{L}\left\{h(x) * f(x)\right\} = \mathcal{L}\left\{h(x)\right\} \mathcal{L}\left\{f(x)\right\} = \\
&= \mathcal{L}\left\{ \sum_{n=-\infty}^{\infty} \frac{a_{-n-1}\, x^{n}}{\Gamma(n+1)} \right\} \mathcal{L}\left\{f(x)\right\} = \\
&= \sum_{n=-\infty}^{\infty} \frac{a_{-n-1}\, \mathcal{L}\left\{x^n\right\}}{\Gamma(n+1)}\, \mathcal{L}\left\{f(x)\right\} = \\
&= \sum_{n=-\infty}^{\infty} \frac{a_{-n-1}}{t^{n+1}}\, \mathcal{L}\left\{f(x)\right\} = \\
&= \sum_{n=-\infty}^{\infty} a_n\, t^n\, \mathcal{L}\left\{f(x)\right\} = \\
&= g(t)\, \mathcal{L}\left\{f(x)\right\} \qquad (9.7)
\end{aligned}$$

since the linearity of the Laplace transform combined with the result of (8.5) shows clearly that

$$
\begin{aligned}
g(D)\, f(x) = \sum_{n=-\infty}^{\infty} a_n\, D^n\, f(x) &= \sum_{n=-\infty}^{\infty} a_n\, \mathcal{L}^{-1} \Big\{ t^n\, \mathcal{L}\left\{ f(x) \right\} \Big\} = \\
&= \mathcal{L}^{-1} \left\{ \sum_{n=-\infty}^{\infty} a_n\, t^n\, \mathcal{L}\left\{ f(x) \right\} \right\} = \\
&= \mathcal{L}^{-1} \Big\{ g(t)\, \mathcal{L}\left\{ f(x) \right\} \Big\}
\end{aligned} \tag{9.8}
$$

as well as (8.11) shows for the Fourier transform that

$$
\begin{aligned}
g(D)\, f(x) = \sum_{n=-\infty}^{\infty} a_n\, D^n\, f(x) &= \sum_{n=-\infty}^{\infty} a_n\, \mathcal{F}^{-1} \Big\{ (it)^n\, \mathcal{F}\left\{ f(x) \right\} \Big\} = \\
&= \mathcal{F}^{-1} \left\{ \sum_{n=-\infty}^{\infty} a_n\, (it)^n\, \mathcal{F}\left\{ f(x) \right\} \right\} = \\
&= \mathcal{F}^{-1} \Big\{ g(it)\, \mathcal{F}\left\{ f(x) \right\} \Big\}
\end{aligned} \tag{9.9}
$$

These are useful tools for the calculation of functions of the derivative. As an example, the following case is considered with the help of Fourier transforms

$$
\begin{aligned}
\cos(D)\, \sin(x) &= \mathcal{F}^{-1} \Big\{ \cos(it)\, \mathcal{F}\left\{ \sin(x) \right\} \Big\} = \\
&= \mathcal{F}^{-1} \Big\{ \cos(it)\, \pi i \big(\delta(t+1) - \delta(t-1) \big) \Big\} = \\
&= \cosh(1)\, \sin(x)
\end{aligned} \tag{9.10}
$$

and directly as in (5.2)

$$
\begin{aligned}
\cos(D)\, \sin(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, D^{2n} \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, (-1)^n \sin(x) = \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n)!} \sin(x) = \cosh(1)\, \sin(x)
\end{aligned} \tag{9.11}
$$

yielding both methods the same result. The case of (5.3) also matches when calculated with Fourier transforms.

10 Cauchy Integral Formula

Another way of generalizing the derivative to non-integer order is given by the Cauchy integral formula that plays a key role in complex analysis,

$$
D^n\, f(z) = \frac{n!}{2\pi i} \oint \frac{f(t)}{(t-z)^{n+1}}\, dt \tag{10.1}
$$

Despite its generalization to any complex number α seems straightforward, it must be taken into account that while being n integer there is an isolated singularity at $t = z$, being it non-integer there is a branch point, what means that the integration contour has to be chosen carefully. Otherwise, the generalization only involves changing the factorial to the gamma function, so defining what is known as the Cauchy-type fractional derivative

$$ {}_{z_0}D_z^{\alpha} f(z) = \frac{\Gamma(\alpha+1)}{2\pi i} \int\limits_{C(z_0,\, z^+)} \frac{f(t)}{(t-z)^{\alpha+1}}\, dt \tag{10.2} $$

where supposing that the branch line starts at $t = z$ and passes through z_0, the contour C starts at $t = z_0$, encircles $t = z$ once in the positive sense and returns to $t = z_0$ where now the integrand has a different value.

It can be proved that this generalization of the derivative is equivalent to the Riemann-Liouville derivative with a lower limit of z_0 for the appropriate values of α in which both derivatives are defined.

11 Properties

Fractional derivatives satisfy quite well all the properties that one could expect from them, despite some of them are only characteristic of integer order differentiation and some other have restrictions. For instance, the property of linearity (1.10) is fulfilled, while that of the iteration (1.9) has some restrictions in the cases that positive differentiation orders are present.

Some properties that include summatories can be generalized changing the summatories into integrals. One such property is the expansion in Taylor series

$$ f(z) = \int_{-\infty}^{\infty} \frac{{}_aD_z^t f(z_0)}{\Gamma(t+1)} (z-z_0)^t \, dt \tag{11.1} $$

and other is the Leibniz rule

$$ {}_aD_z^{\alpha} f(z) g(z) = \int_{-\infty}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(t+1)\,\Gamma(\alpha-t+1)}\, {}_aD_z^{\alpha-t} f(z)\, {}_aD_z^t g(z)\, dt \tag{11.2} $$

These and other generalized properties can be applied to the study of special functions, which often can be expressed in terms of simple formulas involving fractional derivatives. For instance, Gauss's hypergeometric function can be expressed as

$$ {}_2F_1(a,b;c;x) = \frac{\Gamma(a)\, x^{1-c}}{\Gamma(b)}\, {}_0D_x^{b-c} \frac{x^{b-1}}{(1-x)^a} \tag{11.3} $$

12 Local Operators

Despite of some interesting results, we have left unanswered important questions about the functions of the derivative. For instance, it would be very interesting to know if integer order derivatives are or not the only kind of functions of the derivative that are local, and if are or not other kind of local operators that are not functions of the derivative.

Looking at (5.2) it seems that in fact the only local functions of the derivative are integer order derivatives and their finite sums, being non-locality the result of adding infinite terms with displacement functions steadily increasing the distance to the point in which the function is evaluated. However, this impediment can be overcome by defining the derivative in the following way

$$D^{\alpha} f(x) = \lim_{a \to x} {}_{a}D_{x}^{\alpha} f(x) \tag{12.1}$$

so that the limit assures locality. In order to avoid another feature of fractional derivatives that sometimes is not wanted -the fact that the derivative of a constant is not zero- it is usual to define the local fractional derivative as

$$D^{\alpha} f(x) = \lim_{y \to x} {}_{x}D_{y}^{\alpha} \big(f(y) - f(x) \big) \tag{12.2}$$

or similar forms. So there can be operators being local and taking an infinite number of evaluations of the function, provided that the points in which these evaluations are carried out -or the value of their displacement function-remains being infinitesimal. The question now is if there are other local operators besides the local fractional derivative and functions of it.

It could be argued that if a function admits its expansion in Taylor series, all the information about the function is indeed in its derivatives, so that any operator giving information about the function will be unnecessary and reducible to a sum of integer order derivatives. However, on the one hand there are functions that are not differentiable but are local fractional differentiable, and on the other hand, even if local operators could be reduced to ordinary derivatives for differentiable functions, that does not assure that ordinary derivatives are the most appropriate local operators for all the tasks.

Characterizing all local operators is important, for locality makes of them tools that could be useful in understanding Nature.